Equations (20) are the quaternion-rate equations. Observe that there are no singularities in these equations and no trigonometric functions. In fact, they are linear differential equations with time-varying coefficients. For specified time histories of p, q, r, Eqs. (20) are solved for ξ, η, ζ, χ and the instantaneous direction cosine matrix relating the moving frame x, y, z to the stationary frame X, Y, Z is then determined by Eq. (19). The results can also be generalized to the case where X, Y, Z has some known motion.

Using the previous results, it is possible now to solve explicitly a problem recently posed by Bar-Itzhack.[3] His problem is to determine initial values of the quaternion components given initial values of the direction cosine matrix relating the moving and fixed frames. His technique involves an iterative procedure, but this is not necessary since a direct solution is possible in the following way.

The Euler parameters ℓ, m, n, θ can be immediately related to the direction cosine elements using Eq. (7). Designate the elements by a_{ij}, then, adding the main diagonal elements produces

$$\cos\theta = (a_{11} + a_{22} + a_{33} - 1)/2 \tag{21a}$$

whereas subtracting the off-diagonal elements in pairs gives

$$\ell = (a_{23} - a_{32})/2\sin\theta$$
$$m = (a_{31} - a_{13})/2\sin\theta,\ n = (a_{12} - a_{21})/2\sin\theta \tag{21b}$$

Now

$$\chi = \cos(\theta/2) = [(1 + \cos\theta)/2]^{1/2} = (1 + a_{11} + a_{22} + a_{33})^{1/2}/2 \tag{22a}$$

Again

$$\xi = \ell\sin(\theta/2) = (a_{23} - a_{32})\sin(\theta/2)/2\sin\theta = (a_{23} - a_{32})/4\chi \tag{22b}$$

and similarly

$$\eta = m\sin(\theta/2) = (a_{31} - a_{13})/4\chi \tag{22c}$$

$$\zeta = n\sin(\theta/2) = (a_{12} - a_{21})/4\chi \tag{22d}$$

Equations (22) determine the quaternion components explicitly in terms of the direction cosines. Thus when initial values for the latter are known, the initial quaternion is determined. Of course, it was not necessary to work through the Euler axis and angle to derive Eqs. (22). Clearly, they also follow directly from the direction cosine matrix (19).

Conclusions

The quaternion scheme for rigid-body attitude determination has been developed starting with the Euler axis and angle. The axis/angle parameters, by themselves, provide a basis for determining attitude [Eqs. (7) and (15)], but they are not as "clean" computationally as the quaternion parameters [Eqs. (19) and (20)]. An explicit solution for the initial quaternion in terms of the initial values of the direction cosines has also been presented.

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Scanning the Celestial Sphere via Open-Loop Magnetic Control

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FOR some satellite missions the capability of scanning the celestial sphere is essential. A scanning attitude behavior can be obtained by active control, or possibly by passive control such as an orbital regression-gravitational torque combination.[1] This Note presents a method of scanning with a spinning symmetric satellite in an equatorial circular orbit equipped with a current coil whose axis is coincident with the satellite spin axis. The interaction of the magnetic moment with the Earth's magnetic field causes precession of the spin axis. In addition, if the scan is initiated at the proper phase angle between the spin axis and the magnetic north pole, and if the current to the coil is actively controlled in the proper open-loop fashion, the precession of the spin axis (with a period of one day) is accompanied by a constant rate increase of the polar angle of the spin axis. The resulting scan pattern on the surface of the unit sphere is that of a "spherical helix," i.e., a linear increase of the polar angle with the angle of precession.

Discussion

The geometry used in the development is shown in Fig. 1. The basis denoted $\{\tilde{X}_\alpha\}$ is an inertial frame of reference with $\tilde{X}_1$ directed toward the first point of Aries and $\tilde{X}_3$ in the direction of the geographic North Pole. The angles λ and σ measure the polar location (elevation) and precession angle of the angular momentum vector $\tilde{H}$, respectively. The angles ψ, θ, and ϕ parametrize the motion of the satellite relative to the angular momentum vector in classical 3,1,3 Euler rotations. The state equations for the rotational motion of the satellite in a convenient form are[2]

$$\dot{h} = M_2\,\mathrm{s}\theta \tag{1}$$

$$\dot{\theta} = (1/h)M_2\,\mathrm{c}\theta \tag{2}$$

$$\dot{\lambda} = (1/h)(M_1\,\mathrm{c}\psi - M_2\,\mathrm{c}\theta\,\mathrm{s}\psi) \tag{3}$$

$$\dot{\sigma} = (1/h\,\mathrm{s}\lambda)(M_1\,\mathrm{s}\psi + M_2\,\mathrm{c}\theta\,\mathrm{c}\psi) \tag{4}$$

$$\dot{\psi} = (h/I) - (1/h)[(\mathrm{ctn}\theta + \mathrm{s}\psi\,\mathrm{ctn}\lambda)M_1 + \mathrm{c}\theta\,\mathrm{c}\psi\,\mathrm{ctn}\lambda M_2] \tag{5}$$

where $s \equiv$ sine, $c \equiv$ cosine, h is the magnitude of the angular momentum, I is the transverse moment of inertia of the satellite, and M_1 and M_2 are external torque components on transverse nonspinning body axes. (The torque component about the spin axis M_3 is zero.)

The components of the Earth's magnetic field (dipole model) at the satellite expressed in the $\{\tilde{X}_\alpha\}$ basis for a circular, equatorial orbit are[3]

$$B_1 = -(B_o/2)\,\mathrm{s}\delta[\mathrm{s}E - 3\,\mathrm{s}(2\omega_o t - E)] \tag{6}$$

$$B_2 = -(B_o/2)\,\mathrm{s}\delta[-\mathrm{c}E + 3\,\mathrm{c}(2\omega_o t - E)] \tag{7}$$

$$B_3 = B_o\,\mathrm{c}\delta \tag{8}$$

where B_o is the field intensity at the orbital altitude, $\delta = 11°$ is the dipole tilt angle, and ω_o is the orbital frequency. The angle $E = \omega_E t + 3\pi/2 - E_o$, where ω_E is the earth rotation rate and the angle E_o is the longitudinal angle at $t = 0$ between the $\tilde{X}_1$ axis and the magnetic North Pole (79°N, 69°W, near Thule, Greenland[4]). The torque produced on the satellite is

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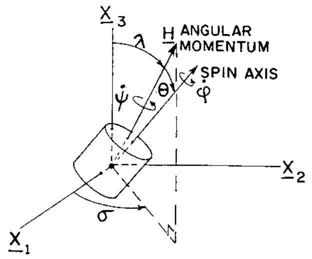

Fig. 1 Angular parametrizations.

$\tilde{M} = \tilde{m} \times \tilde{B}$ where $\tilde{m} = m\tilde{e}_s$ is the magnetic moment vector directed along the spin axis $\tilde{e}_s$ and $\tilde{B}$ is the Earth's field vector. The components of the torque are

$$M_1 = -m \sum_{j=1}^{3} \beta_j B_j \tag{9}$$

$$M_2 = m \sum_{j=1}^{3} \alpha_j B_j \tag{10}$$

where the direction cosines α_j and β_j are given by

$$\alpha_1 = \mathrm{c}\psi\ \mathrm{c}\lambda\ \mathrm{c}\sigma - \mathrm{s}\psi\ \mathrm{s}\sigma$$

$$\alpha_2 = \mathrm{c}\psi\ \mathrm{c}\lambda\ \mathrm{s}\sigma + \mathrm{s}\psi\ \mathrm{c}\sigma$$

$$\alpha_3 = -\mathrm{s}\lambda\ \mathrm{c}\psi$$

$$\beta_1 = -\mathrm{c}\theta\ \mathrm{s}\psi\ \mathrm{c}\lambda\ \mathrm{c}\sigma - \mathrm{s}\sigma\ \mathrm{c}\theta\ \mathrm{c}\psi + \mathrm{s}\theta\ \mathrm{s}\lambda\ \mathrm{c}\sigma$$

$$\beta_2 = -\mathrm{c}\theta\ \mathrm{s}\psi\ \mathrm{c}\lambda\ \mathrm{s}\sigma + \mathrm{c}\theta\ \mathrm{c}\psi\ \mathrm{c}\sigma + \mathrm{s}\theta\ \mathrm{s}\lambda\ \mathrm{s}\sigma$$

$$\beta_3 = \mathrm{c}\theta\ \mathrm{s}\psi\ \mathrm{s}\lambda + \mathrm{s}\theta\ \mathrm{c}\lambda$$

If the ratio of the magnitude of the external torques to the angular momentum is identified as a small parameter, i.e., $\epsilon = (M_1^2 + M_2^2)^{1/2}/h$, Eqs. (1–5) reveal the "slow" variables h, λ, σ, θ and the "fast" variable ψ. In accordance with the method of averaging of Volosov,[5] when $\epsilon = 0$, (Euler-Poinsot motion) $\dot{\psi} = h/I =$ constant, h, λ, σ, θ are constant, and with constant $\dot{\psi}$ inserted into Eqs. (1–4), the equations for the average motion of the satellite are $h =$ constant, $\theta =$ constant and

$$\dot{\lambda} = (m\ \mathrm{c}\theta/h)[B_1\ \mathrm{s}\sigma - B_2\ \mathrm{c}\sigma] \tag{11}$$

$$\dot{\sigma} = (m\ \mathrm{c}\theta/h\ \mathrm{s}\lambda)[B_1\ \mathrm{c}\lambda\ \mathrm{c}\sigma + B_2\ \mathrm{c}\lambda\ \mathrm{s}\sigma - B_3\ \mathrm{s}\lambda] \tag{12}$$

The error involved in the use of these averaged equations is of the order ϵ for time varying up to the order $1/\epsilon$. Explicit time variation is present in Eqs. (11) and (12), and in the time-varying magnetic field components given in Eqs. (6–8). Since magnetic control is practical for near Earth orbits only the orbital rate is higher than the Earth rotation rate and Eqs. (11) and (12) along with B_1 and B_2 may be averaged over the orbital motion. Performing this average gives the average motion of the satellite per orbit which is described by

$$\dot{\lambda} = (-mB_o\ \mathrm{c}\theta\ \mathrm{s}\delta/2h)\ \mathrm{c}(E - \sigma) \tag{13}$$

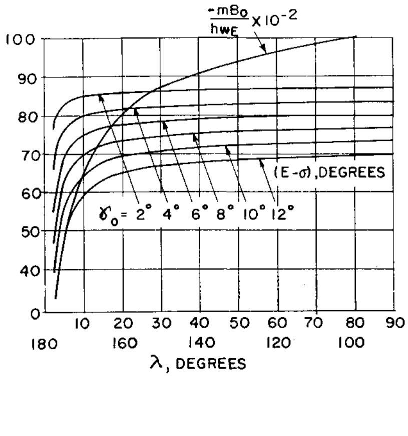

Fig. 2 Phase angle and magnetic moment requirements.

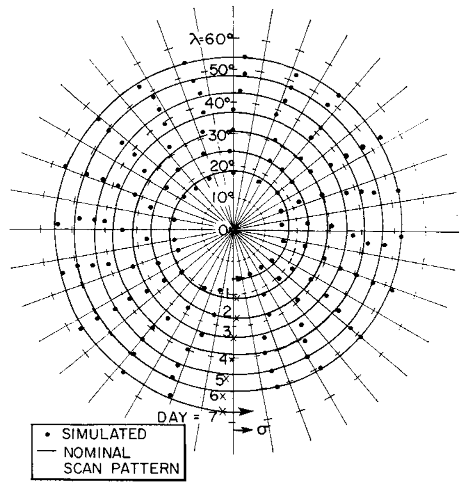

Fig. 3 Simulated scan.

$$\dot{\sigma} = (-mB_o\ \mathrm{c}\theta/2\mathrm{h}\ \mathrm{s}\lambda)[\mathrm{s}\delta\ \mathrm{c}\lambda\ \mathrm{s}(E - \sigma) + 2\ \mathrm{c}\delta\ \mathrm{s}\lambda] \tag{14}$$

A solution for Eqs. (13) and (14) which provides a useful scanning trajectory is one for which the angular momentum vector precesses about the orbit pole (North Pole) at the constant rate of the earth's rotation, i.e., $\dot{\sigma} = \omega_E$, while the polar angle λ increases at a constant rate of γ_o degrees per precession cycle. Thus we seek the solution

$$\dot{\lambda} = (\gamma_o/2\pi)\dot{\sigma} \tag{15}$$

Placing this value of $\dot{\lambda}$ in Eq. (13) and dividing the result by Eq. (14) gives the transcendental equation

$$\gamma_o/2\pi = \mathrm{s}\delta\ \mathrm{s}\lambda\ \mathrm{c}(E - \sigma)/[\mathrm{s}\delta\ \mathrm{c}\lambda\ \mathrm{s}(E - \sigma) + 2\ \mathrm{c}\delta\ \mathrm{s}\lambda] \tag{16}$$

The desired solution of this equation for the phase angle between the angular momentum vector and the magnetic field is

$$(E - \sigma) = \sin^{-1}[\gamma_o\ \mathrm{ctn}\delta/K\pi] - \nu \tag{17}$$

where $K = [1 + (\gamma_o\ \mathrm{ctn}\lambda/2\pi)^2]^{1/2}$ and $tan\nu = -2\pi\ \tan\lambda/\gamma_o$. With this phase angle the magnetic moment required can be found from either Eq. (13) or Eq. (14), e.g., from Eq. (13),

$$m = -\gamma_o h\omega_E/\pi B_o\ \mathrm{c}\theta\ \mathrm{s}\delta\ \mathrm{c}(E - \sigma) \tag{18}$$

where Eq. (15) has been used for $\dot{\lambda}$. Thus if the scan is started at the proper phase angle $(E - \sigma) = 3\pi/2 - E_o$ given by Eq. (17) and if current through the control coil is maintained according to Eq. (18) to give the proper m ($m = inA$ for a coil of n turns with current i and area A), a scan trajectory described by $\lambda = \gamma_o/2\pi\dot{\sigma}t + \lambda(0)$ with $\dot{\sigma} = \omega_E$ will be obtained.

The required phase angle is shown as a function of the polar angle λ in Fig. 2 for $\gamma_o = 2, 4, 6, 8, 10$, and $12°$. The required phase angle increases from zero for scans starting at $\lambda = 0$ and approaches values less than 90° for scans starting at large values of λ. As λ approaches $\pi/2$, the constant K becomes unity, ν approaches $\pi/2$, and from Eq. (17), $(E - \sigma) = \pi/2 - \gamma_o/\pi\ \mathrm{ctn}\delta$ which is the limiting value of the required phase angle. Also shown in Fig. 2 is the required magnetic moment [Eq. (18)] in the form of the dimensionless parameter $-mB_o/h\omega_E$ with the initial angle between the angular momentum vector and spin axis θ assumed to be zero and with B_o expressed in gauss. The single curve for this parameter essentially applies for all of the values of γ_o. Its maximum value is 10^4, which corresponds to an 18-amp turn requirement for a satellite having a spin moment of inertia of 150 kg-m^2, a spin rate of 1 rpm, and a coil radius of 1m in a 600-naut-mile orbit. The control concept has also been studied for use on a 1500-m diam radio telescope satellite in a 6000-km orbit,[6] which, in contrast, requires only 8.6 amp turns because of the large coil area.

A polar plot of the path of the spin axis for a simulated scan starting at an initial polar angle of 15° with a 6° increase per day is shown in Fig. 3. The equations used to simulate the scan were Eqs. (11) and (12), which include the orbital time variations (600-naut-mile orbit) in the magnetic field components. This scanning trajectory was obtained by starting the coil current at the proper initial phase angle, $(E - \sigma) = 76.6°$ (Fig. 2), and by supplying the required coil current in an open-loop fashion according to Eq. (18). The coil current required at any time during the scan was calculated from the time lapsed from scan commencement. That is, at any time t, $\lambda(t) = \gamma_o/2\pi\, t + \lambda(0)$, which gives the required magnetic moment m from Eq. (18) at time t after Eq. (17) has been used to find the nominal phase angle at time t. Thus, in practice, the control system can be mechanized with a clock to provide the time lapsed from the beginning of the scan along with the knowledge of the initial polar angle.

The simulated points in Fig. 3 are the location of the spin axis at $\frac{1}{20}$-day intervals after scan initiation. Hence, if the scan proceeded exactly on the desired nominal pattern, these points would appear at 18° precession angle intervals. The deviation from the nominal pattern is caused by the orbital variation in the magnetic field which was removed by averaging in determining the nominal control. Studies of the effects of error sources for the system, such as the magnetic field model, indicate that the polar angle behavior is more sensitive to error than the precession angle. For example, if the simulation for the conditions applying to Fig. 3 is repeated with an actual dipole tilt of 20° while the control current is based on 11°, the precession angle still increases very nearly at one revolution per day while the polar angle oscillates with an amplitude of ~9° while increasing at 9° per day. As with all open-loop systems, the addition of feedback control may be required to overcome the orbital and other disturbances if close adherence to the nominal scan pattern is desired.

The torque caused by the gravitational field can also be counteracted by the control system. The gravity-gradient torque is given by $\bar{T} = 3\omega_o^2\bar{\xi} \times I\cdot\bar{\xi}$ where $\bar{\xi}$ is the outward geocentric unit vector at the satellite and I is the satellite moment of inertia dyadic. If the components of this torque are included in the state equations [Eqs. (1–5)] along with the magnetic control torques and the equations for the average motion are formulated, the equations, comparable to Eqs. (13) and (14), which result are

$$\dot{\lambda} = (-mB_o\, c\theta\, s\delta/2h)\, c(E - \sigma) \tag{19}$$

$$\dot{\sigma} = \frac{-mB_o\, c\theta}{2h\, s\lambda}[s\delta\, c\lambda\, s(E - \sigma) + 2\, c\delta\, s\lambda] + \frac{3\omega_o^2(I_3 - I)\, c\lambda}{4h}[1 - 3\, c^2\theta] \tag{20}$$

with h and θ constant. The only change resulting from the gravitational torque is the last term in Eq. (20), which represents the average effect per orbit of the gravitational torque on the spin axis precession rate $\dot{\sigma}$. In this term I_3 is the spin axis moment of inertia and I is the transverse moment of inertia. If the scan pattern described by $\lambda(t) = (\gamma_o/2\pi)\dot{\sigma}t + \lambda(0)$ with $\dot{\sigma} = \omega_E$ is again desired, and if the scan is initiated with $\theta = 0$, the phase angle requirement is

$$(E - \sigma) = \sin^{-1}[\gamma_o\, \text{ctn}\delta/C\pi] - \alpha \tag{21}$$

where

$$C = [(1 + b)^2 + (\gamma_o\, \text{ctn}\lambda/2\pi)^2]^{1/2}$$

$$\tan\alpha = -2\pi(1 + b)\tan\lambda/\gamma_o$$

and

$$b = 3\omega_o^2(I_3 - I)\, c\lambda/2h\omega_E$$

The magnetic moment required can be found from either Eq. (19) or Eq. (20) using the value of $(E - \sigma)$ from Eq. (21). Thus, the control system can also be designed to provide the scan while counteracting the averaged gravitational torque.

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Parameter Sensitivity of Aerodynamic Coefficients Determined from Experimental Data

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Nomenclature

$C_{m\alpha}$	= restoring moment coefficient due to angle of attack, rad^{-1}
$C_{mp\beta}$	= magnus moment coefficient due to angle of attack and roll rate, rad^{-2}
C_{mq}	= damping moment coefficient due to inertial angular rates, rad^{-1}
d	= body diameter, ft
E	= noise or error variation in the data, rad
I_x	= roll moment of inertia about body x axis, slug-ft^2
I	= lateral moment of inertia, slug-ft^2
$M_{p\beta},M_q$	= magnus and damping moments, ft-lb-sec/rad^2
M_α	= restoring moment, ft-lb/rad
p	= roll angular velocity, rad/sec
q'	= dynamic pressure, psf
S	= body cross-sectional area, ft^2
t	= time, sec
V	= total free-stream velocity, fps
$\bar{\alpha},\bar{\beta}$	= angles of attack and sideslip, rad
ζ	= $pI_x/2I$, rad/sec
$\lambda_{1,2}$	= damping exponent, sec^{-1}
ξ	= complex total angle of attack, rad; subscripts a and d indicate analytical solution and data, respectively
Φ	= performance index
$\omega_{1,2}$	= nutation and precession frequencies, rad/sec

Introduction

AERODYNAMIC force and moment coefficients for missiles are commonly determined by performing wind-tunnel or free-flight tests and using analytical techniques to extract the coefficients. Unrealistic values may result because of lack of sensitivity of the analytical solution in the test range to changes in the coefficients and to data noise. Severe problems can arise in analyzing resonant motion, because the behavior is very sensitive to the coefficient values. This Note presents an analytical concept for predicting the effectiveness of data

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